On the Force required to Stop a Moving Electrified Sphere.

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§ 1. *Energy of a Charged Sphere in Steady Motion: Heaviside's Method.*—When the velocity of a system of electric charges is changed in any manner, waves of disturbance travel out from the system in all directions with velocity v, the speed of light, and carry both energy and momentum* to the distant parts of the electro-magnetic field. Dr. Oliver Heaviside† was the first to notice that these waves might be made to yield important information as to energy and momentum of the moving system. Thus, when a charged sphere is in steady motion the electro-magnetic field possesses electric energy U, magnetic energy T and momentum M. If, now, the sphere be suddenly brought to rest, the forces which hold it at rest do no work, since their points of application do not move, and hence the total energy in the field is the same after the sphere is brought to rest as it was before. When the sphere is stopped, a pulse with depth equal to the diameter of the sphere travels outwards. The electric and magnetic forces in this pulse ultimately vary inversely as the distance and hence the energy and the momentum in the pulse tend to constant values as the pulse travels out to infinity. Outside the pulse the electric and magnetic forces are the same as if the sphere had continued in steady motion and therefore are, after an infinite time, inversely proportional to the square of the distance from the point which the centre of the sphere would have reached had its motion continued. Hence the energy and momentum outside the pulse ultimately vanish. On the other hand, the field within the spherical region bounded by the pulse is the same as that due to the sphere at rest and hence the energy in this part of the field is ultimately U_s, the electrostatic energy of the sphere at rest. Thus, if W be the limiting value of the energy in the pulse, the energy in the field is ultimately U_s+W. But, before the sphere was stopped, the energy was $U+T$. Hence,

$$U+T = U_s+W. \tag{1}$$

Dr. Heaviside calculated W for the pulse due to a sphere with a surface-charge and so found $U+T$, his value for $U+T$ agreeing with that which I

* The momentum per unit volume is $(\mu K/4\pi)\,\mathbf{VEH}$, where $\mathbf{VEH}$ is the vector product of the electric and magnetic forces.

† 'The Electrician,' November 29, 1901, p. 210. See also Searle, 'Phil. Mag.,' January, 1907, p. 131.

had previously found by a direct integration of the energy in the field of a moving sphere, the integration extending through all space.*

Dr. Paul Hertz† has applied this method to a sphere with a uniform volume-charge and has shown that the energy and momentum of the sphere in steady motion are six-fifths of the energy and momentum of a sphere of the same radius with an equal surface-charge. He has also indicated how the method may be applied to any system whatever, and I have shown how to carry out the details of calculation in the cases of a charged ellipsoid of revolution moving along its axis and of a charged disc moving in its own plane.‡

§ 2. *Force required to Start a Charged Sphere: Paul Hertz's Method.*—But this list does not by any means exhaust the possibilities of the method of pulses.§ For Dr. Hertz has applied it to determine the force which must be applied to a sphere with either a volume- or a surface-charge to cause the sphere to start suddenly from rest and then to move uniformly along a straight line with any velocity u not greater than v, the velocity of light.

If F be the required force, the work done by F when the sphere has been in motion for a time t is

$$\int_0^t \mathrm{F}u\,dt.$$

Before the sphere began to move the energy in the field was U_s. After the time t it is

$$\mathrm{U}_s+\int_0^t \mathrm{F}u\,dt.$$

If at the time t the sphere be suddenly brought to rest, the stopping forces do no work and hence the energy in the field remains unchanged. After an infinite time the energy in the field consists of U_s together with W', the energy in the compound pulse formed by the pair of pulses generated by starting and stopping the sphere.

Hence
$$\mathrm{U}_s+\mathrm{W}' = \mathrm{U}_s+\int_0^t \mathrm{F}u\,dt,$$

or
$$\mathrm{F} = \frac{1}{u}\frac{d\mathrm{W}'}{dt}. \tag{2}$$

It must be noticed that the two pulses are not concentric, since their centres are separated by the distance ut. In the case of a sphere with a surface-charge, the pulses destroy each other's effects where they overlap, since the electric and magnetic forces caused by the sudden stopping of the sphere are

* 'Phil. Mag.,' Oct., 1897.

† "Untersuchungen über unstetige Bewegungen eines Electrons," 'Inaugural Dissertation,' Göttingen, 1904, p. 49. See also Searle, 'Phil. Mag.,' January, 1907, p. 132.

‡ 'Phil. Mag.,' January, 1907.

§ 'Dissertation,' p. 65.

equal in magnitude and opposite in direction to those caused by the starting of the sphere, and are also constant throughout the depth of each pulse. For a sphere with a volume-charge, E and H are not constant throughout the depth of each pulse and the calculation then becomes a little more complicated.

It is necessary to distinguish three stages in the process of starting the sphere. If a be the radius of the sphere, the first stage lasts from $t = 0$ to $t = 2a/(v+u)$, the second from $t = 2a/(v+u)$ to $t = 2a/(v-u)$ and the third from $t = 2a/(v-u)$ onwards. If F_1, F_2 and F_3 denote the forces required in the three stages and if k stand for $1-u/v$, Dr. Hertz's results are as follows:—

Sphere with Uniform Surface-charge Q.

$$F_1 = \frac{Q^2}{2Ka^2}\left\{\frac{v}{u}-\frac{v^2-u^2}{2u^2}\log\frac{v+u}{v-u}\right\}, \tag{3}$$

$$F_2 = \frac{Q^2}{2Ka^2}\left\{\frac{1}{4}+\frac{v}{2u}-\frac{3v^2}{4u^2}+\frac{2av}{u^2t}-\frac{a^2}{u^2t^2}+\frac{v^2-u^2}{2u^2}\log\frac{(v-u)\,t}{2a}\right\},$$

$$F_3 = 0.$$

Thus, in the first stage the force is constant, while in the third stage it is zero.

When u tends to equality with v, the expressions tend to the following limits:—

$$F_1 = \frac{Q^2}{2Ka^2}, \qquad t < a/v,$$

$$F_2 = \frac{Q^2}{2Ka^2}\left\{\frac{2a}{vt}-\frac{a^2}{v^2t^2}\right\} \qquad t > a/v.$$

There is no third stage now, since the second stage extends from $t = a/v$ to infinite values of the time.

Sphere with Uniform Volume-charge Q.

$$F_1 = \frac{Q^2u}{8Ka^6v}\left\{16a^3vt-12a^2v^2t^2+\left(1+\frac{3u^2}{5v^2}\right)v^4t^4\right\}, \tag{4}$$

$$\begin{aligned} F_2 = \frac{Q^2u^2}{8Ka^6v^2}\Big\{ & -\frac{4a^6}{v^2t^2}+\frac{48a^5}{5vt}+(-18k+9k^2)\,a^4 \\ & +(12k^2-8k^3)\,a^3vt+\left(-3k^3+\frac{9k^4}{4}\right)a^2v^2t^2 \\ & +\left(\frac{3k^5}{40}-\frac{k^6}{16}\right)v^4t^4\Big\}, \end{aligned}$$

$$F_3 = 0.$$

When $u = v$, so that $k = 0$, the expressions become

$$F_1 = \frac{Q^2}{8Ka^6}(16a^3vt - 12a^2v^2t^2 + \tfrac{8}{5}v^4t^4), \qquad t < a/v, \tag{4a}$$

$$F_2 = \frac{Q^2}{8Ka^6}\left(-\frac{4a^6}{v^2t^2} + \frac{48a^5}{5vt}\right). \qquad t > a/v.$$

§ 3. *Force required to stop a Moving System. Pulse Method.*—I find that, if Dr. Hertz's work be slightly extended, the force required to suddenly stop a charged system is easily calculated. For the sake of simplicity, the investigation will be limited to the case in which the momentum of the system in steady motion, as well as the momentum in the pulse formed when it is stopped, are parallel to the direction of motion. If F be the force which must be applied to the system at any time t, after it has been brought to rest at $t = 0$, the positive direction of F being opposite to that of u, then F is also the force which the electro-magnetic field exerts on the system in the direction of u.

The momentum given up by the electro-magnetic field from $t = 0$ to $t = t$ is

$$\int_0^t F dt.$$

During this period the force F does no work, since the system is at rest, and hence the energy of the system is unchanged during this period.

At the time t let the system be restarted with the same velocity u without change of direction, and let G be the force which must be applied to the system at any subsequent time in the direction of u in order to maintain the velocity u. This force lasts from $t = t$ to $t = t'$, where $t'-t$ is determined by the condition that in the time $t'-t$ the pulse formed on restarting the system has completely passed over the system. When $u = v$, the time $t'-t$ is infinite. During the interval $t'-t$, the momentum of the system is increased by

$$\int_t^{t'} G dt,$$

and hence the total gain of momentum is

$$\int_t^{t'} G dt - \int_0^t F dt.$$

During the interval $t'-t$, the energy of the system has been increased by

$$u\int_t^{t'} G dt.$$

The stopping and the restarting of the system each give rise to a pulse, and the compound pulse so formed carries off energy W′ and momentum P′.

Before the system was stopped the energy of the electro-magnetic field was $U+T$ and its momentum was M, and at an infinite time after the stopping and restarting the energy is $U+T+W'$ and the momentum is $M+P'$, since the energy and momentum in the parts of the field outside the compound pulse ultimately vanish.

Equating the two expressions for the gain of momentum, we have

$$\int_t^{t'} \mathrm{G}dt - \int_0 \mathrm{F}dt = \mathrm{P}'.$$

Similarly,

$$u\int_t^{t'} \mathrm{G}dt = \mathrm{W}'.$$

Hence

$$\int_0^t \mathrm{F}dt = \mathrm{W}'/u - \mathrm{P}', \tag{5}$$

and thus we find that the force required to stop the system is given by

$$\mathrm{F} = \frac{d}{dt}\left(\frac{\mathrm{W}'}{u} - \mathrm{P}'\right). \tag{6}$$

This force will become zero as soon as $W'/u-P'$ becomes constant, which will occur as soon as t is so great that the two pulses due to the stopping and restarting do not overlap.

Since the system is restarted from the position in which it was stopped, the two pulses are concentric, though their radii differ by vt, for one pulse is formed at $t = 0$ and the other at the time t. The fact that the pulses are concentric introduces an element of simplicity into the calculations.

The whole momentum I, given up by the electro-magnetic field during the time for which the stopping force lasts, is found by taking t so great that the two pulses do not overlap. In this case each pulse has the same energy and the same momentum, and hence, if W and P be the energy and the momentum in each of the separate pulses, $W' = 2W$ and $P' = 2P$.

Hence

$$\mathrm{I} = \int_0^t \mathrm{F}dt = 2\left(\frac{\mathrm{W}}{u} - \mathrm{P}\right). \tag{7}$$

§ 4. *Force required to stop a Sphere with Surface-charge.*—We may now apply the results of § 3 to find the force required to stop a sphere of radius a with a uniform surface-charge Q. If W_0 be the energy and P_0 the momentum sent out in the pulse formed when the velocity u is suddenly destroyed, we have*

$$\mathrm{W}_0 = \frac{\mathrm{Q}^2}{2\mathrm{K}a}\left\{\frac{v}{u}\log\frac{v+u}{v-u} - 2\right\}. \tag{8}$$

$$\mathrm{P}_0 = \frac{\mathrm{Q}^2}{2\mathrm{K}au}\left\{\left(\frac{3v}{2u} - \frac{u}{2v}\right)\log\frac{v+u}{v-u} - 3\right\}. \tag{9}$$

* See Searle, 'Phil. Mag.,' Jan., 1907, pp. 131, 132.

Hence $$\frac{W_0}{u}-P_0=\frac{Q^2}{2Kav}\left\{\frac{v}{u}-\frac{v^2-u^2}{2u^2}\log\frac{v+u}{v-u}\right\}. \tag{10}$$

If the time t be less than $2a/v$, the two pulses will overlap over a depth $2a-vt$, and in this part the electric and magnetic forces due to one pulse will exactly neutralise those due to the other, since these forces are constant throughout the depth of either pulse. There remain two shells, each of depth vt, where the forces do not cancel. The total energy W' and the total momentum P' in the compound pulse are therefore given by

$$W'=2vt\frac{W_0}{2a},\qquad P'=2vt\frac{P_0}{2a}.$$

Hence, if F_0 be the force required to stop the sphere with a surface-charge, we have, by (6),

$$\begin{aligned}F_0&=\frac{v}{a}\left(\frac{W_0}{u}-P_0\right)\\&=\frac{Q^2}{2Ka^2}\left\{\frac{v}{u}-\frac{v^2-u^2}{2u^2}\log\frac{v+u}{v-u}\right\}.\end{aligned} \tag{11}$$

Hence the force is constant during the time $2a/v$ for which it acts. The impulse, I_0, of this force, or the momentum which the electro-magnetic field gives up to the agent stopping the sphere, is $2aF_0/v$.

Hence $$I_0=\frac{Q^2}{Kav}\left\{\frac{v}{u}-\frac{v^2-u^2}{2u^2}\log\frac{v+u}{v-u}\right\}. \tag{12}$$

When u tends to equality with v, so that the initial velocity of the sphere becomes more and more nearly equal to the velocity of light, the expression for the force tends to a limit. For as $v-u$ tends to zero, $(v-u)\log(v-u)$ tends to the limit zero, and hence F_0 tends to the limit

$$F_0=\frac{Q^2}{2Ka^2}. \tag{13}$$

The formula (11) gives us no information as to the value of F_0 when u is equal to v, since the expression then becomes indeterminate. But, instead of deducing the force F_0 from the energy and momentum in the compound pulse, we can (as in § 6) calculate its value, when $u=v$, by direct integration of E_x over the surface of the sphere, where E_x is the component, parallel to the direction of u, of the electric force in the pulse. We then find that the value of F_0 for $u=v$ is identical with the limit $Q^2/2Ka^2$ of the general value of the force. Hence, there is no discontinuity in the force when $u=v$.

When u/v is very small, we have, by (11),

$$F_0=\frac{Q^2u}{3Ka^2v}. \tag{14}$$

§ 5. *Force required to Stop a Sphere with Volume-charge.*—When the sphere of radius a has a uniform volume-charge Q, the calculation of the force required to stop the sphere is a little more difficult, because the electric and magnetic forces are not constant throughout the depth of the pulse formed on suddenly starting or stopping the sphere. If we take two parallel planes at distances z and $z+dz$ from the surface of the sphere, and if $d\mathrm{Q}$ be the charge between them,

$$\frac{d\mathrm{Q}}{dz} = \frac{3\mathrm{Q}z(2a-z)}{4a^3}.$$

As H. A. Lorentz* and Paul Hertz† have pointed out, at a great time after the formation of one of these pulses the electric and magnetic forces at a point on the radius normal to these planes, and at a distance z from the outer surface of the pulse, are proportional to $d\mathrm{Q}/dz$.

There are two parts of the compound pulse where the separate pulses due to the stopping and the restarting of the sphere do not overlap, and in each of these parts, of depth vt, there are equal amounts of energy and of momentum. Now, when a pulse is generated by starting or stopping an equal sphere with a surface-charge Q, the energy per unit depth of pulse is $\mathrm{W}_0/2a$, where W_0 is given by (8), while for this case $d\mathrm{Q}/dz = \mathrm{Q}/2a$. Hence, if W_1 be the energy in the part of the compound pulse where the separate pulses do not overlap,

$$\begin{aligned} \mathrm{W}_1 &= 2\frac{\mathrm{W}_0}{2a}\int_0^{vt}\left(\frac{d\mathrm{Q}}{dz}\right)^2\left(\frac{2a}{\mathrm{Q}}\right)^2 dz \\ &= \frac{\mathrm{W}_0}{a}\int_0^{vt}\left\{\frac{3z(2a-z)}{2a^2}\right\}^2 dz \\ &= \frac{3\mathrm{W}_0 v^2t^2}{8a^5}\{8a^2vt-6av^2t^2+\tfrac{6}{5}v^3t^3\}. \end{aligned}$$

In that part of the compound pulse where the two separate pulses overlap, the effective value of $d\mathrm{Q}/dz$ is

$$\frac{3\mathrm{Q}z(2a-z)}{4a^3}-\frac{3\mathrm{Q}(z-vt)(2a-z+vt)}{4a^3},$$

or

$$\frac{3\mathrm{Q}vt}{4a^3}(2a+vt-2z),$$

since the electric and magnetic forces have opposite directions in the two separate pulses, and since the outer surfaces of the two pulses are separated by the distance vt. This part of the compound pulse extends from $z = vt$ to $z = 2a$, and hence, if W_2 be the energy in this part, we have

* 'Encyklopädie der Mathematischen Wissenschaften,' "Electronentheorie," p. 188.

† 'Dissertation,' p. 36. See also Searle, 'Phil. Mag.,' January, 1907, p. 123.

$$\begin{aligned}W_2 &= \frac{W_0}{2a}\int_{vt}^{2a}\left(\frac{dQ}{dz}\right)^2\left(\frac{2a}{Q}\right)^2 dz \\ &= \frac{W_0}{2a}\int_{vt}^{2a}\left\{\frac{3vt\,(2a+vt-2z)}{2a^2}\right\}^2 dz \\ &= \frac{3W_0v^2t^2}{8a^5}(2a-vt)^3.\end{aligned}$$

Hence, if W′ be the energy in the compound pulse,

$$W' = W_1+W_2 = \frac{3W_0v^2t^2}{8a^5}(8a^3-4a^2vt+\tfrac{1}{5}v^3t^3). \tag{15}$$

Similarly, if P_0 be the momentum in the pulse formed on starting a sphere with a surface-charge, where P_0 is given by (9), the momentum in the compound pulse is given by

$$P' = \frac{3P_0v^2t^2}{8a^5}(8a^3-4a^2vt+\tfrac{1}{5}v^3t^3). \tag{16}$$

We can now find the force required to stop the sphere, for we have, by (6),

$$\begin{aligned}F &= \frac{d}{dt}\left(\frac{W'}{u}-P'\right) \\ &= \frac{3v}{8a^5}(16a^3vt-12a^2v^2t^2+v^4t^4)\left(\frac{W_0}{u}-P_0\right) \\ &= \frac{3v}{8a^5}vt\,(2a-vt)^2(4a+vt)\left(\frac{W_0}{u}-P_0\right) \\ &= \frac{3Q^2}{16Ka^6}(16a^3vt-12a^2v^2t^2+v^4t^4)\left(\frac{v}{u}-\frac{v^2-u^2}{2u^2}\log\frac{v+u}{v-u}\right).\end{aligned} \tag{17}$$

If I be the total momentum given up by the electro-magnetic field when the sphere is stopped, we have, by (7),

$$I = 2\,(W/u-P).$$

But, as Dr. Hertz* has shown, $W = \frac{6}{5}W_0$ and $P = \frac{6}{5}P_0$, and, hence, $I = \frac{6}{5}I_0$, where I_0 is given by (12).

The same result follows if we integrate the expression (17) with respect to the time from $t = 0$ to $t = 2a/v$, for

$$\int_0^{2a/v} F\,dt = \tfrac{6}{5}\,.\,2\left(\frac{W_0}{u}-P_0\right) = \tfrac{6}{5}I_0.$$

When u tends to equality with v, we see, by (17), that F tends to the limit

$$F = \frac{3Q^2}{16Ka^6}(16a^3vt-12a^2v^2t^2+v^4t^4). \tag{18}$$

* 'Dissertation,' p. 49. See also Searle, 'Phil. Mag.,' January, 1907, p. 132.

It is proved by another method in § 7 that the value of F for $u = v$ is equal to the limit (18).

When u^2/v^2 is very small, we have, by (10),

$$\frac{W_0}{u} - P_0 = \frac{Q^2 u}{3Kav^2},$$

and then (17) becomes

$$F = \frac{Q^2 u}{8Ka^6 v}(16a^3vt - 12a^2v^2t^2 + v^4t^4). \tag{19}$$

If F_0 be the force required to stop the sphere with a surface-charge, so that F_0 is given by (11), we can write (17) in the form

$$F = \tfrac{3}{8}a^{-4}(16a^3vt - 12a^2v^2t^2 + v^4t^4)\,F_0. \tag{20}$$

The maximum value of F occurs when $vt = a(\sqrt{3}-1)$ and the maximum value is

$$F_{\max} = 9(\sqrt{3} - \tfrac{3}{2})\,F_0 = 0{\cdot}209\,F_0.$$

§ 6. *Force required to Stop a Sphere with Surface-charge when $u = v$.*—We now pass on to calculate by a direct method the force required to stop a sphere with a surface-charge, when the initial velocity of the sphere is equal to that of light. We now find the force experienced by each element of the charge at a time t after the sphere has been brought to rest, and then integrate over the surface of the sphere.

When an elementary charge dQ is suddenly stopped, the electric force in the pulse is given by*

$$dE = \frac{u \sin\gamma}{Kr(v - u\cos\gamma)}\,\frac{dQ}{p}. \tag{21}$$

Here $r = vt$ is the distance from the point where the charge is stopped and γ is the angle between the radius r, drawn *from* that point, and the direction of the initial velocity u. Further, p is the infinitesimal width of the charge measured in a direction parallel to r. The electric force dE is in the plane of r and u, and is at right angles to r, while it has a *positive* component in the direction of u.

If dE_x be the component of dE in the direction of u, we find that, when $u = v$,

$$dE_x = dE\sin\gamma = \frac{dQ\,(1+\cos\gamma)}{Krp}. \tag{22}$$

Let O (fig. 1) be the centre of the sphere after it has been brought to rest and let OA be the direction of its initial velocity. Let P be a point on the surface of the sphere and let POA $= \theta$. About P as centre describe two

* See Searle, 'Phil. Mag.,' January, 1907, p. 121. The expression was first given by Heaviside, 'The Electrician,' October 11, 1901.

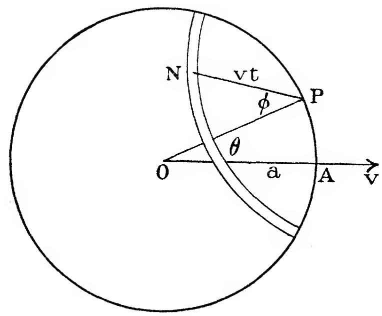

Fig. 1.

spheres of radii r and $r+dr$, cutting the charged sphere in two circles. Let N be a point on the band bounded by these circles, let NPO $= \phi$, and let the plane NPO be inclined at an angle ψ to the plane POA.

Since $r = 2a\cos\phi$, we have $dr = -2a\sin\phi d\phi$. But the angle subtended at O by the width of the band is twice the angle subtended at P, and is, therefore, $-2d\phi$, and thus the width of the band is $-2ad\phi$ or $dr/\sin\phi$. If the charge on that element of the ring which is defined by the planes ψ and $\psi+d\psi$ be $d\mathrm{Q}$, then

$$d\mathrm{Q} = \frac{\mathrm{Q}}{4\pi a^2}(-2ad\phi)\, r\sin\phi d\psi,$$

while
$$p = dr = -2a\sin\phi d\phi.$$

Hence
$$\frac{d\mathrm{Q}}{p} = \frac{\mathrm{Q}rd\psi}{4\pi a^2},$$

and thus, by (22),
$$d\mathrm{E}_x = \frac{\mathrm{Q}(1+\cos\gamma)\, d\psi}{4\pi\mathrm{K}a^2}. \tag{23}$$

It is only the band defined by r, where $r = vt$, which acts upon P by its pulse at the time t, and hence we obtain the complete value of the part of E_x which is due to the action of pulses by integrating (23) with respect to ψ.

Now r cos γ is the projection of NP (*not* PN) upon OA, and thus we easily find—

$$\cos\gamma = \cos\theta\cos\phi + \sin\theta\sin\phi\cos\psi. \tag{24}$$

Hence
$$\mathrm{E}_x = \int_0^{2\pi}\frac{d\mathrm{E}_x}{d\psi}\, d\psi = \frac{\mathrm{Q}(1+\cos\theta\cos\phi)}{2\mathrm{K}a^2};$$

but $\cos\phi = r/2a = vt/2a$, and thus

$$\mathrm{E}_x = \frac{\mathrm{Q}}{2\mathrm{K}a^2}\left(1+\frac{vt\cos\theta}{2a}\right). \tag{25}$$

The element of charge at P is also acted on by the charges on the sphere whose distances from P are less than vt, as well as by those whose distances

are greater than vt. The elements within the distance vt act on the element at P according to the ordinary electrostatic law, and the element at P acts on them in the same way. Hence the actions between the element at P and the elements within the distance vt are in equilibrium, and may be left out of account in estimating the force exerted on the sphere as a whole. The elements outside the distance vt act on the element at P in the same way as if they had continued to move on with the speed of light. The electric forces due to these elements are therefore at right angles to the direction of u, and therefore contribute nothing to the total force.

Hence, if F_0 be the total force experienced by the sphere at time t, we can find F_0 by intergrating (25) over the surface of the sphere. Thus—

$$F_0 = \int_0^\pi \frac{E_x Q}{4\pi a^2} 2\pi a^2 \sin\theta d\theta = \frac{Q^2}{4Ka^2}\int_0^\pi \left(1+\frac{vt\cos\theta}{2a}\right)\sin\theta d\theta = \frac{Q^2}{2Ka^2}. \quad (26)$$

This value is equal to the limit (13) to which (11), the general expression for the force, tends as u tends to equality with v. The force experienced by the sphere has this value for the time $2a/v$ during which the pulse is passing over the sphere. As soon as the pulse is clear of the sphere the force vanishes.

§ 7. *Force required to Stop a Sphere with Volume-charge when $u = v$.*—The method of § 6 may be applied to find the force required to suddenly stop a sphere with a uniform volume-charge, when the initial velocity is equal to that of light.

In fig. 2, O is the centre of the sphere after it has been brought to rest, OA is the direction of its original velocity, and P is any point within the

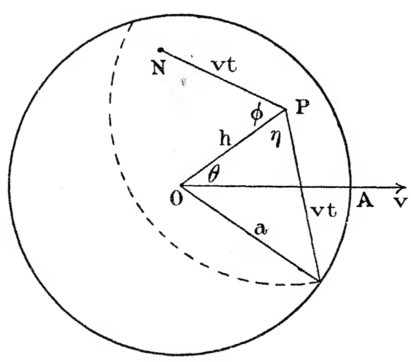

Fig. 2.

sphere. Let OP $= h$ and POA $= \theta$, and let OA be taken as the axis of x. Let N be a point in the sphere at a distance $r = vt$ from P, let NPO $= \phi$, and let the plane NPO be inclined at an angle ψ to the plane POA.

The charge per unit volume is $3Q/4\pi a^3$, and thus the x-component of the electric force at P due to the pulse arising from stopping the charge (of thickness $p = dr$) included in the elementary volume $r^2 \sin\phi d\phi d\psi dr$ is, by (22),

$$dE_x = \frac{3Qr^2 \sin\phi d\phi d\psi dr}{4\pi a^3 . dr} \cdot \frac{1+\cos\gamma}{Kr},$$

where $\cos\gamma$ is given by (24).

Since $r = vt$, we have

$$dE_x = \frac{3Qvt}{4\pi a^3 K} \sin\phi\,(1+\cos\gamma)\,d\phi d\psi.$$

The x-component of the electric force at P at time t due to pulses is obtained by integrating this expression over that part of the surface of the sphere of centre P, and of radius vt, which lies within the charged sphere. For the reasons explained in § 6 the force experienced by the *whole* sphere is equal to that due to the action of the pulses alone.

When the sphere of radius vt lies entirely within the charged sphere, ψ goes from 0 to 2π and ϕ from 0 to π. In this case we have

$$\begin{aligned} E_x &= \frac{3Qvt}{4\pi a^3 K}\int_0^\pi\int_0^{2\pi} \sin\phi\,(1+\cos\gamma)\,d\phi d\psi \\ &= \frac{3Qvt}{2a^3 K}\int_0^\pi \sin\phi\,(1+\cos\theta\cos\phi)\,d\phi \\ &= \frac{3Qvt}{a^3 K}. \end{aligned} \tag{27}$$

When the sphere of radius vt lies partly outside the charged sphere, ψ goes from 0 to 2π, while ϕ goes from 0 to η, where η is the angle subtended at P by a radius drawn from O to a point on the circle of intersection of the two spheres. From the triangle having this radius as base and P as vertex we have

$$\cos\eta = \frac{h^2+v^2t^2-a^2}{2hvt} \tag{28}$$

In this case we have

$$\begin{aligned} E_x &= \frac{3Qvt}{2a^3 K}\int_0^\eta \sin\phi\,(1+\cos\theta\cos\phi)\,d\phi \\ &= \frac{3Qvt}{2a^3 K}\{1-\cos\eta+\tfrac{1}{2}\cos\theta\,(1-\cos^2\eta)\}. \end{aligned} \tag{29}$$

The time of passage of the whole pulse across the sphere may be divided into two stages. In the first stage, which lasts from $t = 0$ to $t = a/v$, some complete spheres of radius vt can be described, but in the second stage, lasting from $t = a/v$ to $t = 2a/v$, the spheres of radius vt are all incomplete.

In the first stage we can describe complete spheres of radius vt about every point within the sphere of centre O and radius $a - vt$.

Now, by (27), E_x has the constant value $3Qvt/a^3K$ throughout the sphere of radius $a - vt$. The charge within that sphere is $Q(a - vt)^3/a^3$, and hence, if X_1 be the force on this part of the charged sphere,

$$X_1 = \frac{3Qvt}{a^3K} \cdot \frac{Q(a-vt)^3}{a^3} = \frac{3Q^2vt(a-vt)^3}{Ka^6}.$$

If X_2 be the force on the spherical shell of radii $a - vt$ and a, then X_2 is due to the action of incomplete spheres, and thus—

$$X_2 = \frac{3Q}{4\pi a^3} \cdot \int_0^\pi \int_{a-vt}^a E_x \,.\, 2\pi h^2 \sin\theta \, d\theta dh,$$

where E_x is now given by (29). When we integrate with respect to θ, the term in E_x involving $1-\cos^2\eta$ vanishes, and thus, substituting for $\cos\eta$ from (28), we find

$$X_2 = \frac{9Q^2vt}{2Ka^6}\int_{a-vt}^a \left(1-\frac{h^2+v^2t^2-a^2}{2hvt}\right)h^2dh$$

$$= \frac{3Q^2v^2t^2}{Ka^6}\left(\tfrac{9}{4}a^2-3avt+\tfrac{51}{48}v^2t^2\right).$$

The resultant force F is equal to X_1+X_2, and thus

$$F = \frac{3Q^2vt}{Ka^6}\left\{(a-vt)^3+\tfrac{9}{4}a^2vt-3av^2t^2+\tfrac{51}{48}v^3t^3\right\}$$

$$= \frac{3Q^2}{16Ka^6}\left\{16a^3vt-12a^2v^2t^2+v^4t^4\right\}. \tag{30}$$

In the second stage from a/v to $2a/v$, no complete sphere of radius vt can be described about any point within the charged sphere. But in this case we need not integrate throughout the whole volume of the sphere, since the pulses due to all the elements of the sphere have now completely passed over the part enclosed by the sphere $h = vt-a$, and have left this part free from any force due to pulses.

Integrating throughout the spherical shell of radii $vt-a$ and a, we find, for the force during the second stage,

$$F = \frac{3Q}{4\pi a^3}\int_0^\pi\int_{vt-a}^a E_x \,.\, 2\pi h^2 \sin\theta d\theta dh,$$

where E_x is given by (29). Thus

$$F = \frac{9Q^2vt}{2Ka^6}\int_{vt-a}^a \left(1-\frac{h^2+v^2t^2-a^2}{2hvt}\right)h^2dh$$

$$= \frac{3Q^2}{16Ka^6}\left\{16a^3vt-12a^2v^2t^2+v^4t^4\right\}. \tag{31}$$

This expression is identical with that given by (30) for the force during the first stage. Hence, the force is expressed by the same formula during the whole time of its action. The value also agrees with the limit (18) approached by (17), the general value of the force, as found in § 5, when u tends to equality with v.

§ 8. *Comparison of Forces required for Starting and Stopping a Charged Sphere.*—It is interesting to compare Dr. Hertz's value for the force required during the first stage of starting the sphere with the value obtained in the present paper for the force required to stop the sphere.

Sphere with a Surface-charge.

From (3) and (11) it will be seen that the stopping force is identical with the starting force during the first stage of the motion. This result could have been foretold without any detailed calculation. For it is easily seen that Dr. Hertz's method leads to a starting force which is constant during the first stage, while my method leads to a constant stopping force. When $t = 0$, the two forces must be equal, since the sphere is at that time in the same position relative to the pulses as it is when it is suddenly stopped. Thus, since each force is constant, they remain equal up to the time $2a/(v+u)$.

Sphere with a Volume-charge.

The method employed in the case of a surface-charge shows that the forces must be equal at $t = 0$. But since dQ/dz is zero at the surface of the sphere, it is easily seen that both the starting and the stopping forces must vanish at $t = 0$. This result also follows from the detailed formulæ (4) and (17), since each expression contains t as a factor.

But when u/v is infinitesimal, the displacement of the sphere during the time $2a/(v+u)$, when it is started, is also infinitesimal, and hence for such a value of u/v the force required during the first stage of the starting of the sphere must be equal to that required to stop it. The latter force lasts for $2a/v$, but, since u/v is infinitesimal, the first stage may be considered to last for the same time. Comparing Dr. Hertz's expression (4) with (19), it will be seen that they agree when u^2/v^2 is negligible in comparison with unity.

The length of the second stage is from $2a/(v+u)$ to $2a/(v-u)$, and thus may be considered as zero.

When $u = v$, it appears from ($4a$) and (31) that, as long as v^2t^2 is negligible compared with a^2, the stopping force is three halves of the starting force.